Constraints on Black Holes in Classical and Semiclassical Theories of Gravity

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Abstract

Constraints on the geometry of a static spherically symmetric black hole are obtained by requiring the spacetime curvature to be analytic at the event horizon. For a zero temperature black hole further constraints are obtained by also requiring that the semiclassical trace equation be satisfied when conformally invariant fields are present. It is found that zero temperature black holes whose sizes lie within a certain range do not exist. The range depends on the numbers and types of conformally invariant quantized fields that are present.

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I. INTRODUCTION

One of the unanswered questions in semiclassical gravity is how quantized fields alter the spacetime geometry near the event horizon of a black hole. This is important because the thermodynamic properties of a black hole are determined by the geometry at the event horizon. Some work has been done to answer this question using linearized semiclassical gravity and either analytical approximations or numerical computations of the stress-energy tensor in Schwarzschild spacetime [1–5]. Further progress has been hampered by problems with the analytical approximations near the event horizons of other black holes [6–8] and by the difficulty involved in numerically computing the stress-energy tensor for quantized fields in black hole spacetimes.

Given these difficulties and the importance which quantum effects may have in black hole spacetimes, it is useful to see if constraints can be placed on the possible geometry's that black holes can have at their event horizons. In this paper constraints relating to static spherically symmetric black holes are found using two methods. The first is very general and applies to both classical and semiclassical metric theories of gravity. It is to assume that the spacetime curvature is analytic at the event horizon. This assures that there will be no curvature singularities there. This is a very reasonable assumption that is satisfied by all known vacuum solutions to Einstein's equations that correspond to black holes. Further it is not without precedent. In one of the uniqueness theorems which Hawking [9] has proven for rotating stationary black holes it is assumed that the metric is analytic at the event horizon. Recently Zaslavskii [10] has determined the general form of the metric near the event horizon of near extreme and extreme charged black holes when the black holes are in a cavity and the grand canonical ensemble is utilized. He assumed that the metric can be expanded in a power series in the radial coordinate r near the event horizon.

The requirement that the curvature be analytic at the event horizon of a static black hole results in significant constraints on the geometry in that region. These are greatest for nonzero temperature black holes where it is found that the spacetime geometry must be of the same general form near the event horizon as that of Schwarzschild and Reissner-Nordström black holes.

The second method involves using the trace equation in semiclassical gravity to constrain the spacetime geometry near the event horizon of a zero temperature black hole. In general it is extremely difficult to solve the semiclassical back reaction equations because of the difficulty in computing the stress-energy tensor for quantized fields in a spacetime with an arbitrary or at least somewhat arbitrary geometry. However, for conformally invariant fields it is well known that the trace of the stress-energy tensor is exactly equal to the trace anomaly which is known analytically in a general spacetime. Since any solution to the semiclassical equations must be a solution to the trace equation, one can put constraints on solutions to the semiclassical equations by examining the trace equation. In general such constraints are not very interesting because they are not very strong. However if the curvature is analytic at the event horizon and if only conformally invariant fields are present, then it is found that there is a range of sizes for which zero temperature black holes cannot exist. The range of excluded sizes depends on the number and types of quantized fields present. In many cases this range extends to zero, resulting in a minimum size for zero temperature black holes.

In what follows we first discuss restrictions which result from requiring the curvature to

be analytic at the event horizon. Then those resulting from the trace equation are given.

II. CONSTRAINTS DUE TO THE CURVATURE

The metric for a static spherically symmetric spacetime can be written in the general form [11]

$$ds^2 = -f(r)dt^2 + \frac{1}{k(r)}dr^2 + r^2 d\Omega^2 \quad . \tag{1}$$

The unique nonvanishing components of the Riemann curvature tensor in an orthonormal frame are

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \frac{v'k}{2} + \frac{vk'}{4} + \frac{v^2 k}{4} \tag{2a}$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = \frac{vk}{2r} \tag{2b}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{k'}{2r} \tag{2c}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1-k}{r^2} \quad , \tag{2d}$$

where $v \equiv f'/f$ and primes denote derivatives with respect to r. If the spacetime has an event horizon then f vanishes on that horizon. The surface gravity at the event horizon is given by the formula

$$\kappa = \frac{v}{2} (fk)^{1/2} \quad . \tag{3}$$

We require that the above components of the Riemann tensor be analytic at the event horizon. From Eq.(2d) it is clear that k must then be analytic at the event horizon. Eq.(2b) shows that the quantity $v\,k$ must also be analytic. This second condition results in the further condition that $k \to 0$ at the event horizon. To see this note that v must diverge at the horizon because $v = f'/f$ and f vanishes there. These conditions can be summarized by saying that near the event horizon v and k have the following leading order behaviors:

$$v = p(r - r_0)^{m-n} \tag{4a}$$

$$k = q(r - r_0)^n \quad . \tag{4b}$$

Here p and q are real positive constants, m and n are integers which satisfy the condition $n > m \geq 0$, and r_0 is the value of r at the event horizon.

Further restrictions can be obtained by considering Eq.(2a). Substituting Eqs.(4a-b) into (2a) one finds that

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -\frac{p\,q}{4} (n - 2m) (r - r_0)^{m-1} + \frac{p^2 q}{4} (r - r_0)^{2m-n} \quad . \tag{5}$$

These terms on the right hand side of (5) must either be separately finite at the horizon or they must cancel. They only cancel if $m = n - 1$ and $p = 2 - n$. However we previously

showed that $n \geq 1$. Thus, since $p > 0$, the only case in which they cancel is $n = 1$, $m = 0$, $p = 1$. In this case it is easy to see that near the horizon $f = c(r - r_0)$ for some positive constant c. If the terms on the right hand side of Eq.(5) vanish separately then the restrictions are $m \geq 1$ and $2m \geq n$.

To summarize, either solutions exist with

$$n = 1, \quad m = 0, \quad p = 1, \tag{6a}$$

or there is the constraint

$$2m \geq n > m \geq 1 \tag{6b}$$

In the first case the surface gravity and hence the temperature is finite and nonzero. In the second case it is always zero. Thus all static spherically symmetric black holes with nonzero temperatures must have

$$f = c(r - r_0) + ...$$
$$k = q(r - r_0) + ... \tag{7}$$

near their event horizons if the components of the Riemann tensor in an orthonormal frame are analytic at the event horizon.

III. CONSTRAINTS DUE TO THE TRACE ANOMALY

Semiclassical gravity can be used to place further constraints on the geometry near the event horizon of a static spherically symmetric black hole in the case that only conformally invariant free quantized fields are present. The semiclassical back reaction equations can be written in the general form

$$G_{\mu\nu} + aA_{\mu\nu} + bB_{\mu\nu} = 8\pi(T_{\mu\nu})_{cl} + 8\pi < T_{\mu\nu} > \tag{8}$$

Here $A_{\mu\nu}$ and $B_{\mu\nu}$ are tensors which result from the variation of a scalar curvature squared term and a Weyl squared term in the gravitational Lagrangian. Their coefficients, a and b, are arbitrary and must in principle be determined by experiment or observation. $(T_{\mu\nu})_{cl}$ is the stress-energy tensor for any classical fields. We shall only be concerned here with the classical electromagnetic field [12]. The trace of $B_{\mu\nu}$ is identically zero as is the trace of the stress-energy tensor for the classical electromagnetic field. The trace of $A_{\mu\nu}$ is equal to $-6\Box R$ [13]. For conformally invariant fields the trace of $< T_{\mu\nu} >$ is equal to the trace anomaly [13]. Thus the trace equation is

$$- R - 6a\Box R = 8\pi[\alpha\Box R + \beta(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2) + \gamma C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}] \tag{9}$$

with

$$\alpha = [N(0) + 6N(1/2) - 18N(1)]/2880\pi^2 \tag{10a}$$
$$\beta = [N(0) + 11N(1/2) + 62N(1)]/2880\pi^2 \tag{10b}$$
$$\gamma = [N(0) + \frac{7}{2}N(1/2) - 13N(1)]/2880\pi^2 . \tag{10c}$$

Here $N(0)$, $N(1/2)$, and $N(1)$ are the number of scalar, four component spin $1/2$, and vector fields respectively.

If Eqs.(4a-b) are substituted into Eq.(9) and condition (6b) is imposed, then the leading order terms near the horizon can be computed for various values of m and n. Consider first the case $2m > n > m \geq 2$. In the limit $r \to r_0$ Eq.(9) becomes

$$-\frac{16\pi}{3r_0{}^2}(\beta + 2\gamma) - \frac{2}{r_0{}^2} = 0 \tag{11}$$

From Eqs.(10a-c) it is clear that for all fields $\beta + 2\gamma > 0$. Thus there are no solutions to the trace equation for values of m and n in this range.

The only other possibility is $2m = n \geq 2$. This means that the only possible zero temperature solutions to the semiclassical back reaction equations have the form

$$v = p(r - r_0)^{-m}$$
$$k = q(r - r_0)^{2m} \quad , \tag{12}$$

near the event horizon, with $m \geq 1$. Substituting (12) into (9) and taking the limit $r \to r_0$ results in the equation

$$\frac{p^2 q}{2} - \frac{\pi\beta p^4 q^2}{3} - \frac{2\pi\gamma p^4 q^2}{3} - \frac{16\pi(\beta + 2\gamma)}{3r_0{}^4} - \frac{2}{r_0{}^2} - \frac{16\pi(\beta - \gamma)p^2 q}{3r_0{}^2} = 0 \quad . \tag{13}$$

This equation has the solutions

$$q_\pm = \frac{1}{4\pi(\beta + 2\gamma)p^2 r_0{}^2} \left[3r_0{}^2 - 32\pi(\beta - \gamma) \pm (768\pi^2\beta^2 - 3072\pi^2\beta\gamma - 288\pi\beta r_0{}^2 + 9r_0{}^4)^{1/2}\right] . \tag{14}$$

It is interesting to look these solutions in the limit $r_0 \to \infty$. They are

$$q_+ \to \frac{3}{2\pi(\beta + 2\gamma)p^2} \tag{15a}$$

$$q_- \to \frac{4}{p^2 r_0{}^2} \tag{15b}$$

The second solution has the same type of behavior as the extreme Reissner-Nordström solution (for which p=2). The first has a different behavior. When substituted into Eqs.(2a-d) it is seen that the spacetime curvature at the event horizon is large for this solution even in the limit $r_0 \to \infty$. Therefore this is not a physically acceptable solution for large r_0 since quantum effects would be large even on macroscopic scales. It should be emphasized here that there are many solutions to the trace equation which are not solutions to the full set of semiclassical back reaction equations.

Finally it is possible to put bounds on the allowed values of r_0 for zero temperature black holes. Since the left hand side of (14) is positive and real, the right hand side must be also. However the right hand side is complex if

$$r_- < r_0 < r_+$$

$$r_\pm = 4(\pi\beta)^{1/2} \left[1 \pm \left(\frac{2}{3\beta}\right)^{1/2}(\beta + 2\gamma)^{1/2}\right]^{1/2} \quad . \tag{16}$$

For all allowed values of β and γ r_+ is real. If $\beta < 4\gamma$ then r_- is imaginary and solutions only occur for $r_0 > r_+$. If $\beta > 4\gamma$ then solutions also occur for $0 < r_0 < r_-$.

We have examined the constraints on static spherically symmetric black holes which are imposed by requiring that the components of the Riemann tensor in an orthonormal frame be analytic at the event horizon. This requirement forces the metric functions for nonzero temperature black holes to be of the form (7). For zero temperature black holes we further imposed the condition that the semiclassical trace equation be satisfied in the case that only conformally invariant fields are present. This resulted in the constraint that zero temperature black holes must have metric functions of the form (12). Also no zero temperature black holes can exist in this case with values of r at the event horizon in the range $r_- < r_0 < r_+$. Here $r_\pm$ is given by Eq.(16) with the understanding that if r_- is complex, the lower limit is zero.

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[11] Throughout we use units such that $\hbar = c = G = k_B = 1$. Our sign conventions are those of C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[12] In fact the results will be valid for any classical stress-energy tensor whose trace vanishes.

[13] See for example N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Freeman, San Francisco, 1973).